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Closed form solutions of Euler–Bernoulli beams with singularities

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Abstract

The problem of the integration of the static governing equations of the uniform Euler–Bernoulli beam with discontinuities is studied. In particular, two types of discontinuities have been considered: flexural stiffness and slope discontinuities. Both the above mentioned discontinuities have been modeled as singularities of the flexural stiffness by means of superimposition of suitable distributions (generalized functions) to a uniform one dimensional field. Closed form solutions of governing differential equation, requiring the knowledge of the boundary conditions only, are proposed, and no continuity conditions are enforced at intermediate cross-sections where discontinuities are shown. The continuity conditions are in fact embedded in the flexural stiffness model and are automatically accounted for by the proposed integration procedure. Finally, the proposed closed form solution for the cases of slope discontinuity is compared with the solution of a beam having an internal hinge with rotational spring reproducing the slope discontinuity.

Keywords: Euler–Bernoulli beam; Singularities; Distribution theory

1. Introduction

Research fields such as fracture mechanics might require study of beams showing singularities along the beam span. Moreover, cases showing abrupt changes of the cross-section or the Young's modulus or the presence of internal constraints in single span beams might result in the appearance of discontinuities in the kinematic solution functions such as curvature and slope functions.

The problem of finding the solution of beams showing physical or geometrical discontinuities along the beam span has been treated in the literature by Yavari et al. (2000, 2001) and Yavari and Sarkani (2001).

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However, in these cases integration is usually performed by seeking continuous solution functions over domains between discontinuities and imposing continuity conditions. Procedures based on integration over the entire beam span, however requiring enforcement of the continuity conditions, have been also proposed in the literature by Falsone (2002). As a result the computational effort depends on the number of discontinuities and no closed form solutions, dependent on boundary conditions only, have been proposed.

However, for the case of singularities in loading conditions only, such as concentrated forces and moments, the adoption of distributions (generalized functions) allowed in the literature the integration of the governing differential without enforcement of additional conditions at cross-sections where singularities appeared (Yavari et al., 2000, 2001; Yavari and Sarkani, 2001; Falsone, 2002). The distribution theory to model singularities in geometrical or physical beam properties have been adopted by Yavari et al. (2001) for both Timoshenko and Euler–Bernoulli beams. They showed that in the case of singularities the governing differential equilibrium equation for Timoshenko beam can always be expressed in terms of a single deflection and single slope functions, while for Euler–Bernoulli beam for uniform beams only, since the product of two generalized functions is not defined in the traditional distribution theory. In any case, whatever solution procedure is adopted, the Laplace transform method (Kanwal, 1983) or the more advantageous auxiliary beam method proposed by Yavari et al., the continuity conditions have to be enforced.

In this paper the problem of the integration of the static differential governing equation of the uniform Euler–Bernoulli beam showing discontinuities is studied. In particular, two types of discontinuities have been considered: flexural stiffness discontinuity and slope discontinuity. Aim of this work is proposing closed form solutions of the governing equation of the beam as functions of the boundary conditions and of the discontinuity intensity able to account for the two types of discontinuities.

The proposed approach is based on the adoption of the distribution (generalized function) theory, where the discontinuities to be considered are accounted for as ad hoc singularities of the flexural stiffness. More precisely, the above mentioned singularities are obtained by modeling the flexural stiffness by means of suitable distributions. In particular flexural stiffness discontinuity is modeled by the unit step function and slope discontinuity is modeled as Dirac's delta distribution appearing in the flexural stiffness function.

It has to be noted that slope discontinuity is usually obtained by means of an internal hinge. Hence, in this paper, the proposed closed form solutions for the above mentioned case is compared with solution of a beam endowed with internal hinge with rotational spring.

Examples of Euler–Bernoulli beams with single discontinuity are presented in order to comment the expected results.

2. Euler–Bernoulli beam with singular flexural stiffness

In this section the static governing equations of the Euler–Bernoulli beam model are recalled and adopted in order to treat the case of beams with variations of flexural stiffness according to singular conditions only, which will be modeled by means of the distribution theory.

The static governing equations of the Euler–Bernoulli beam model are written as follows:

$$V^{\mathrm{I}}(x) = -q(x); \quad M^{\mathrm{I}}(x) = V(x) \tag{1a,b}$$

$$\chi(x) = \frac{M(x)}{E(x)I(x)} \tag{1c}$$

$$\chi(x) = \varphi^{\mathrm{I}}(x); \quad \varphi(x) = -u^{\mathrm{I}}(x) \tag{1d,e}$$

where $q(x)$ is the external load, $V(x)$ and $M(x)$ are the shear force and the bending moment, respectively, $u(x)$, $\varphi(x)$ and $\chi(x)$ are the deflection, slope and curvature functions, respectively, and the prime denotes differentiation with respect to the spatial coordinate x spanning from 0 to the length l of the beam.

The sets of differential Eqs. (1a,b) and (1d,e) represent the equilibrium and compatibility equations, respectively, while the algebraic Eq. (1c) is the constitutive equation relating curvature and bending moment through the spatial variable flexural stiffness $E(x)I(x)$ defined by means the Young's modulus $E(x)$ and the inertia moment $I(x)$.

Combining the compatibility and constitutive equations given by Eq. ((1c)–(e)) yields to the following second order differential equation relating the bending moment with the second derivative of deflection:

$$E(x)I(x)u^{\mathrm{II}}(x) = -M(x) \tag{2}$$

Accounting for the equilibrium equations also, given by Eq. (1a,b), yields to the Euler–Bernoulli fourth order differential governing equation, in terms of deflection only, as follows:

$$[E(x)I(x)u^{\mathrm{II}}(x)]^{\mathrm{II}} = q(x) \tag{3}$$

where the spatial variability of the flexural stiffness has to be accounted for.

Integration of Eq. (2) is usually performed for statically determinate beams in view of the knowledge of the bending moment $M(x)$ through the equilibrium equations, otherwise the more general fourth order differential Eq. (3) has to be integrated. In this work integration of Eq. (3) will be performed for two cases of flexural stiffness, however the second order differential Eq. (2) will be helpful in order to clarify the properties of the assumed flexural stiffness functions.

Let us consider, in particular, the cases of uniform Euler–Bernoulli beams with flexural stiffness presenting singularities according to the following form:

$$E(x)I(x) = E_0I_0[1 - \gamma D(x - x_0)] \tag{4}$$

The flexural stiffness singularity, representing a decrement of intensity γ at abscissa $0 < x_0 < l$ of the constant flexural stiffness E_0I_0, has been modeled by means of a distribution centered at x_0, also called generalized function in the literature, here indicated as $D(x - x_0)$. In what follows two cases of distributions, able to reproduce physical circumstances for the Euler–Bernoulli beam, will be considered. More precisely, the first case to be treated is

$$D(x - x_0) = U(x - x_0) \tag{5}$$

where $U(x - x_0)$ indicates the well known unit step distribution, also known in the literature as Heaviside's function. Substitution of Eq. (5) into Eq. (4) provides the following form for the flexural stiffness of the beam:

$$E(x)I(x) = E_0I_0[1 - \gamma U(x - x_0)] \tag{6}$$

representing a beam model with abrupt variation of the cross-section or of the Young's modulus, resulting in a discontinuous flexural stiffness at the abscissa x_0 and constant elsewhere.

The second case to be analyzed is concerning with the choice:

$$D(x - x_0) = \delta(x - x_0) \tag{7}$$

where $\delta(x - x_0)$ is the Dirac's delta distribution centered at x_0. Substitution of Eq. (7) into Eq. (4) provides the following form for the flexural stiffness of the beam:

$$E(x)I(x) = E_0I_0[1 - \gamma\delta(x - x_0)] \tag{8}$$

representing a beam model whose flexural stiffness is given by a constant value E_0I_0 with the superimposition of a Dirac's delta distribution. Interpretation of such a flexural stiffness is not as straightforward as the previous case. However, substitution of Eq. (8) into the governing equation (3) and double integration lead to the following equation:

$$u^{\mathrm{II}}(x) = \frac{b_1 + b_2 x + q^{[2]}(x)}{E_0 I_0} + \gamma u^{\mathrm{II}}(x)\delta(x - x_0) \tag{9}$$

b_1 and b_2 being integration constants and $q^{[k]}(x)$ a function evaluated as a primitive of order k of the external load function $q(x)$. In particular $q^{[2]}(x)$ appearing in Eq. (9) is expected to be continuous even for discontinuous and concentrated vertical loads, except for load cases presenting concentrated moments (containing a doublet distribution) leading to the presence of a discontinuity in $q^{[2]}(x)$.

In order to infer a description of the curvature function $\chi(x) = -u^{\mathrm{II}}(x)$ of the beam, as a consequence of the flexural stiffness given by Eq. (8), both sides of Eq. (9) are multiplied by $\delta(x - x_0)$ as follows:

$$u^{\mathrm{II}}(x)\delta(x - x_0) = \frac{b_1 + b_2 x + q^{[2]}(x)}{E_0 I_0}\delta(x - x_0) + \gamma u^{\mathrm{II}}(x)\delta(x - x_0)\delta(x - x_0) \tag{10}$$

The first term on the right hand side of Eq. (10) can be considered as a Dirac's delta distribution since $q^{[2]}(x)$ is either a continuous function or, for those cases arising from the presence of concentrated external moments, the discontinuities are assumed in the region $[0, x_0^-] \cup [x_0^+, l]$, hence never coincident with x_0.

Rather, attention has to be devoted to the second term on the right hand side of Eq. (10) in which, under the assumption of the associative property for products of distributions, the product of two Dirac's deltas appears. Definition of the product of Dirac's deltas is still an open question in the mathematical literature. In order to give Eq. (10) some mathematical meaning a theory allowing the product definition of at least two Dirac's deltas has to be adopted. In this paper the following definition of the product of two Dirac's deltas proposed by Bagarello (1995, 2002), described in Appendix A, is adopted:

$$\delta(x - x_0)\delta(x - x_0) = A\delta(x - x_0) \tag{11}$$

where A is a positive constant for which the value 2.013 has been adopted (Appendix A).

In view of Eq. (11), Eq. (10) leads to the following relationship:

$$u^{\mathrm{II}}(x)\delta(x - x_0) = \frac{b_1 + b_2 x + q^{[2]}(x)}{E_0 I_0}\delta(x - x_0) + \gamma A u^{\mathrm{II}}(x)\delta(x - x_0) \tag{12}$$

and after simple algebra the following expression can be obtained:

$$u^{\mathrm{II}}(x)\delta(x - x_0) = \frac{1}{1 - \gamma A}\frac{b_1 + b_2 x + q^{[2]}(x)}{E_0 I_0}\delta(x - x_0) \tag{13}$$

Finally, substitution of Eq. (13) into Eq. (9) gives the following explicit expression of the curvature for the considered beam model:

$$\chi(x) = -u^{\mathrm{II}}(x) = -\frac{b_1 + b_2 x + q^{[2]}(x)}{E_0 I_0}\left(1 + \frac{\gamma}{1 - \gamma A}\delta(x - x_0)\right) \tag{14}$$

Eq. (14) suggests a curvature function given as the superimposition of a Dirac's delta distribution, centered at x_0, to the function $(b_1 + b_2 x + q^{[2]}(x))/E_0 I_0$, surely continuous at x_0. As a consequence the model under study is concerning with a slope function $\varphi(x)$, primitive of $\chi(x)$ according to Eq. (1d), presenting a discontinuity at x_0; hence the choice of the flexural stiffness given in Eq. (8) allows the treatment of the case of a beam with an internal hinge at the abscissa x_0.

3. Euler–Bernoulli beam with jump discontinuities in flexural stiffness

The case of flexural stiffness provided by Eq. (6) presenting a jump discontinuity at x_0 and constant for $x \neq x_0$ is considered in this section by means of the superimposition of a unit step function $D(x-x_0) = U(x-x_0)$ to the uniform flexural stiffness E_0I_0. The governing equation (3) assumes the following form:

$$[E_0I_0(1-\gamma U(x-x_0))u^{\mathrm{II}}(x)]^{\mathrm{II}} = q(x) \tag{15}$$

It has to be noted that, for the case under study, the physical constraint of non-negativity for the flexural stiffness requires the condition $\gamma \leqslant 1$ for the discontinuity intensity.

A double integration of differential equation (15) leads to

$$u^{\mathrm{II}}(x) = \frac{1}{E_0I_0(1-\gamma U(x-x_0))}(b_1 + b_2x + q^{[2]}(x)) \tag{16}$$

where b_1 and b_2 are integration constants. After simple algebra Eq. (16) can be rewritten as follows:

$$\chi(x) = -u^{\mathrm{II}}(x) = -\left(2c_3 + 6c_4x + \frac{q^{[2]}(x)}{E_0I_0}\right)\left(1 + \frac{\gamma}{1-\gamma}U(x-x_0)\right) \tag{17}$$

where the following positions have been accounted for

$$c_3 = \frac{b_1}{2E_0I_0}; \quad c_4 = \frac{b_2}{6E_0I_0} \tag{18a,b}$$

The first integration of Eq. (17) provides the slope function as follows, where properties of distributions have been accounted for

$$\begin{aligned}\varphi(x) &= -u^{\mathrm{I}}(x)\\ &= -c_2 - 2c_3\left[x + \frac{\gamma}{1-\gamma}(x-x_0)U(x-x_0)\right] - 3c_4\left[x^2 + \frac{\gamma}{1-\gamma}\left(x^2 - x_0^2\right)U(x-x_0)\right] - \frac{q^{[3]}(x)}{E_0I_0}\\ &\quad - \frac{\gamma}{1-\gamma}\frac{q^{[3]}(x) - q^{[3]}(x_0)}{E_0I_0}U(x-x_0)\end{aligned} \tag{19}$$

and the subsequent integration provides the following closed form expression for the deflection function:

$$\begin{aligned}u(x) &= c_1 + c_2x + c_3\left[x^2 + \frac{\gamma}{1-\gamma}(x-x_0)^2U(x-x_0)\right] + c_4\left[x^3 + \frac{\gamma}{1-\gamma}\left(x^3 - 3x_0^2x + 2x_0^3\right)U(x-x_0)\right]\\ &\quad + \frac{q^{[4]}(x)}{E_0I_0} + \frac{\gamma}{1-\gamma}\frac{q^{[4]}(x) - q^{[4]}(x_0) - q^{[3]}(x_0)(x-x_0)}{E_0I_0}U(x-x_0)\end{aligned} \tag{20}$$

In Eqs. (19) and (20) c_1 and c_2 are further integration constants. The integration constants c_1, c_2, c_3, c_4 can be obtained by means of enforcement of boundary conditions.

It has to be noted that the solution functions, in view of the discontinuity of the flexural stiffness, show continuous deflection and slope functions but to a curvature function showing a discontinuity at x_0.

Multiplication of the curvature function given by Eq. (17) by the flexural stiffness showing the jump discontinuity given by Eq. (6) provides the following bending moment expression:

$$M(x) = E(x)I(x)\chi(x) = -E_0I_0\left(2c_3 + 6c_4x + \frac{q^{[2]}(x)}{E_0I_0}\right) \tag{21}$$

Differentiation of Eq. (21) leads to the shear force function as follows:

$$V(x) = M^{\mathrm{I}}(x) = -E_0 I_0 \left(6c_4 + \frac{q^{[1]}(x)}{E_0 I_0} \right) \tag{22}$$

Eqs. (21) and (22) provide bending moment and shear force as continuous functions, except for possible discontinuities contained by $q^{[1]}(x)$ and $q^{[2]}(x)$ functions due to the external loads, as expected. Moreover, it has to be noted that the γ constant representing the discontinuity intensity of the flexural stiffness does not explicitly appear in Eqs. (21) and (22) but its influence is contained in the c_3, c_4 constants by means of enforcement of the boundary conditions. Hence, on the basis of the proposed integration procedure, it has been possible to transfer the influence of the jump discontinuities of the flexural stiffness to the boundary conditions, at least as far as the bending moment and shear force are concerned. For statically determinate beams, since $M(x)$ and $V(x)$ are independent of the flexural stiffness, the c_3 and c_4 constants are expected to be independent of γ and x_0.

4. Euler–Bernoulli beam with slope discontinuities

In this section the case of flexural stiffness provided by Eq. (8) with the adoption of a Dirac's delta distribution $D(x - x_0) = \delta(x - x_0)$ at x_0 is considered. The governing equation (3) assumes the following form:

$$[E_0 I_0 (1 - \gamma\delta(x - x_0)) u^{\mathrm{II}}(x)]^{\mathrm{II}} = q(x) \tag{23}$$

According to the integration procedure presented in Appendix B, the solution of the fourth order governing differential Eq. (23) can be written as follows:

$$u(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + d_1(x) + d_2(x)x + d_3(x)x^2 + d_4(x)x^3 \tag{24}$$

being c_1, c_2, c_3, c_4 integration constants and $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$ functions that can be evaluated as solution of first order differential Eq. (A2.14).

Integration of Eq. (A2.14) is performed by means of integration by parts only, leading to

$$\begin{aligned} d_1(x) = & -\frac{q^{[1]}(x)x^3 - 3q^{[2]}(x)x^2 + 6q^{[3]}(x)x - 6q^{[4]}(x)}{6E_0 I_0} - \gamma \int u^{\mathrm{II}}(x) x \delta(x - x_0)\,\mathrm{d}x \\ & - \frac{\gamma}{6}\left[(u^{\mathrm{III}}(x)x - 3u^{\mathrm{II}}(x))x^2 \delta(x - x_0) + u^{\mathrm{II}}(x)x^3 \delta^{\mathrm{I}}(x - x_0) \right] \end{aligned} \tag{25a}$$

$$\begin{aligned} d_2(x) = & \frac{q^{[1]}(x)x^2 - 2q^{[2]}(x)x + 2q^{[3]}(x)}{2E_0 I_0} + \gamma \int u^{\mathrm{II}}(x) \delta(x - x_0)\,\mathrm{d}x \\ & + \frac{\gamma}{2}\left[(u^{\mathrm{III}}(x)x - 2u^{\mathrm{II}}(x))x \delta(x - x_0) + u^{\mathrm{II}}(x)x^2 \delta^{\mathrm{I}}(x - x_0) \right] \end{aligned} \tag{25b}$$

$$d_3(x) = -\frac{q^{[1]}(x)x - q^{[2]}(x)}{2E_0 I_0} - \frac{\gamma}{2}\left[(u^{\mathrm{III}}(x)x - u^{\mathrm{II}}(x))\delta(x - x_0) + u^{\mathrm{II}}(x)x\delta^{\mathrm{I}}(x - x_0) \right] \tag{25c}$$

$$d_4(x) = \frac{q^{[1]}(x)}{6E_0 I_0} + \frac{\gamma}{6}[u^{\mathrm{III}}(x)\delta(x - x_0) + u^{\mathrm{II}}(x)\delta^{\mathrm{I}}(x - x_0)] \tag{25d}$$

where $\delta^{\mathrm{I}}(x - x_0)$ is the first distributional derivative of the Dirac's delta distribution, also called doublet distribution.

Application of integration by parts to Eq. (A2.14) left unsolved in Eq. (25) the integrals $\int u^{\mathrm{II}}(x) x \delta(x - x_0)\,\mathrm{d}x$ and $\int u^{\mathrm{II}}(x)\delta(x - x_0)\,\mathrm{d}x$. However, according to Eq. (13), the product $u^{\mathrm{II}}(x)\delta(x - x_0)$

is coincident with a Dirac's delta centered at x_0, hence, by applying the usual rules of distributions, the following expressions holds (Guelfand and Chilov, 1972; Hoskins, 1979; Lighthill, 1958; Zemanian, 1965):

$$\int u^{\mathrm{II}}(x)x\delta(x-x_0)\,\mathrm{d}x = \frac{1}{1-\gamma A}\left(2c_3 + 6c_4x_0 + \frac{q^{[2]}(x_0)}{E_0I_0}\right)x_0U(x-x_0) \tag{26a}$$

$$\int u^{\mathrm{II}}(x)\delta(x-x_0)\,\mathrm{d}x = \frac{1}{1-\gamma A}\left(2c_3 + 6c_4x_0 + \frac{q^{[2]}(x_0)}{E_0I_0}\right)U(x-x_0) \tag{26b}$$

where Eq. (18) have been accounted for.

Substitution of Eqs. (25) and (26) into Eq. (24) provides the following closed form expression for the deflection function:

$$\begin{aligned} u(x) = {} & c_1 + c_2x + c_3\left[x^2 + 2\frac{\gamma}{1-\gamma A}(x-x_0)U(x-x_0)\right] + c_4\left[x^3 + 6\frac{\gamma}{1-\gamma A}x_0(x-x_0)U(x-x_0)\right] \\ & + \frac{q^{[4]}(x)}{E_0I_0} + \frac{\gamma}{1-\gamma A}\frac{q^{[2]}(x_0)(x-x_0)}{E_0I_0}U(x-x_0) \end{aligned} \tag{27}$$

where the constants c_1, c_2, c_3, c_4 can be obtained by means of enforcement of boundary conditions.

The function given by Eq. (27) is the sought single continuous deflection function, for the case under study, leading to the following discontinuous slope function:

$$\begin{aligned} \varphi(x) &= -u^{\mathrm{I}}(x) \\ &= -c_2 - 2c_3\left[x + \frac{\gamma}{1-\gamma A}U(x-x_0)\right] - 3c_4\left[x^2 + 2\frac{\gamma}{1-\gamma A}x_0U(x-x_0)\right] \\ &\quad - \frac{q^{[3]}(x)}{E_0I_0} - \frac{\gamma}{1-\gamma A}\frac{q^{[2]}(x_0)}{E_0I_0}U(x-x_0) \end{aligned} \tag{28}$$

A further differentiation of Eq. (28) provides the closed form expression for the curvature function:

$$\chi(x) = -u^{\mathrm{II}}(x) = -\left(2c_3 + 6c_4x + \frac{q^{[2]}(x)}{E_0I_0}\right)\left[1 + \frac{\gamma}{1-\gamma A}\delta(x-x_0)\right] \tag{29}$$

Bending moment function is obtained by multiplying the curvature function given by Eq. (29) by the flexural stiffness given by Eq. (8) as follows:

$$M(x) = E(x)I(x)\chi(x) = -E_0I_0\left(2c_3 + 6c_4x + \frac{q^{[2]}(x)}{E_0I_0}\right) \tag{30}$$

where Eq. (11) has been accounted for. Shear force function is obtained by means of differentiation of Eq. (30) as follows:

$$V(x) = M^{\mathrm{I}}(x) = -E_0I_0\left(6c_4 + \frac{q^{[1]}(x)}{E_0I_0}\right) \tag{31}$$

Eqs. (30) and (31) provide bending moment and shear force as continuous functions, except for possible discontinuities contained in $q^{[1]}(x)$ and $q^{[2]}(x)$ functions due to the external loads, as expected.

It has to be noted that Eqs. (30) and (31) coincide formally with Eqs. (21) and (22), the difference appearing in the values of c_3 and c_4 affected by the singularity intensity γ of the flexural stiffness, which does not appear explicitly in Eqs. (30) and (31), by means of enforcement of the boundary conditions. However, for statically determinate beams, the flexural stiffness singularity does not affect the bending moment and the shear force, hence the c_3 and c_4 constants appearing in Eqs. (21) and (22) are expected to be coincident with those of Eqs. (30) and (31) and independent of γ.

The slope function (28) presents a jump discontinuity $\Delta\varphi(x_0)$ at x_0 which is explicitly evaluated as follows:

$$\Delta\varphi(x_0) = \varphi(x_0^+) - \varphi(x_0^-) = -\frac{\gamma}{1-\gamma A}\left(2c_3 + 6c_4x_0 + \frac{q^{[2]}(x_0)}{E_0I_0}\right) \tag{32}$$

being x_0^+ and x_0^- the abscissas to the right and to the left of the position x_0 where the discontinuity is located. A comparison of Eq. (32) with the bending moment given by Eq. (30) evaluated at x_0 leads to

$$\Delta\varphi(x_0) = \frac{\gamma}{1-\gamma A}\frac{M(x_0)}{E_0I_0} \tag{33}$$

Eq. (33) corresponds to the presence of an internal hinge at x_0 endowed with rotational spring with stiffness k_φ given as

$$k_\varphi = \frac{1-\gamma A}{\gamma}E_0I_0 \tag{34}$$

An inspection of Eq. (34) shows that for $\gamma = 1/A$ an internal hinge with no rotational spring is recovered. For $\gamma = 0$ Eq. (34) provides an infinite stiffness k_φ, representing the absence of any discontinuity for the flexural stiffness and the uniform Euler–Bernoulli beam is recovered. For $\gamma < 0$ Eq. (34) provides a negative stiffness k_φ which does not possess any physical meaning. It can hence be concluded that for the case under study the discontinuity intensity γ is subject to the $0 \leqslant \gamma \leqslant 1/A$ constraint.

5. Applications

Applications of the closed form solutions presented in Sections 3 and 4 are here presented and discussed for the clamped–clamped beam, subjected to a uniform vertical load q, depicted in Fig. 1. In particular the following data have been assumed:

$$l = 500\ \text{cm};\quad E_0 = 2.1\times10^4\ \text{kN/cm}^2;\quad I_0 = 5224\ \text{cm}^4;\quad q = 0.015\ \text{kN/cm}$$

5.1. Along beam discontinuity

In this section the results concerning the two different types of discontinuities, treated in this paper, located at $0 < x_0 < l$ are reported. Cases of boundary discontinuity, located at $x_0 = 0$ or $x_0 = l$, deserve particular attention and will be discussed in the next section. In particular the position $x_0 = 300\,\text{cm}$ of the discontinuity is assumed.

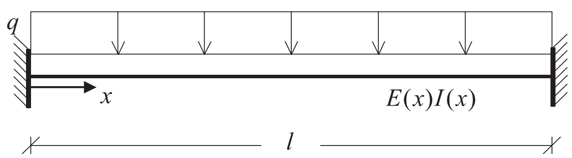

Fig. 1. Clamped–clamped beam under study.

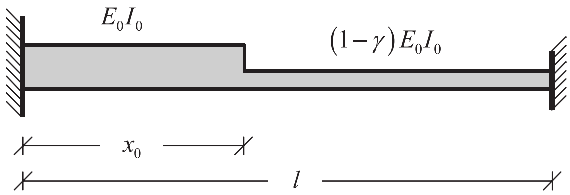

Fig. 2. Clamped–clamped beam with flexural stiffness discontinuity.

5.1.1. Flexural stiffness discontinuity

The beam shows an abrupt flexural stiffness change at x_0 from the value E_0I_0 to the value $(1-\gamma)E_0I_0$, as depicted in Fig. 2, and $\gamma = 0.5$ has been assumed.

For the beam under study the integration constants c_1, c_2, c_3, c_4, evaluated by imposing the boundary conditions, are

$$c_1 = 0; \quad c_2 = 0; \quad c_3 = \frac{q}{24E_0I_0}\frac{l^6 + \gamma^2 x_0^6 - \gamma x_0^2 l^2 (9l^2 - 16x_0 l + 9x_0^2)}{l^4 + \gamma^2 x_0^4 - 2\gamma x_0 l(2l^2 - 3x_0 l + 2x_0^2)} \tag{35a,b,c}$$

$$c_4 = -\frac{q}{12E_0I_0}\frac{l^5 + \gamma^2 x_0^5 - \gamma x_0 l(3l^3 - 2x_0 l^2 - 2x_0^2 l + 3x_0^3)}{l^4 + \gamma^2 x_0^4 - 2\gamma x_0 l(2l^2 - 3x_0 l + 2x_0^2)} \tag{35d}$$

The closed form solutions in terms of deflection, slope and curvature functions, obtained in Section 3 and given by Eqs. (20), (19), (17), respectively, are plotted in Fig. 3 and compared with the solution of the uniform beam with constant flexural stiffness E_0I_0. Fig. 3 show the expected continuity of deflection and slope functions and the discontinuity of the curvature as a consequence of the first order discontinuity of the slope function clearly shown by Fig. 3b. The abrupt decrement of the flexural stiffness results in a general increment of the deflection and slope functions with respect to the uniform beam.

In Fig. 4 bending moment and shear force functions, obtained by making use of Eqs. (21) and (22), have been plotted, showing some difference with the uniform beam since the beam under study is statically indeterminate and the c_3 and c_4 constants depend on x_0 and γ.

It has to be noted that by substituting the integration constants c_1, c_2, c_3, c_4 into the deflection function given by Eq. (20) and making the limit for $\gamma \to -\infty$ we obtain:

$$u(x) = \frac{q}{24E_0I_0}(x - x_0)^2 x^2 [1 - U(x - x_0)] \tag{36}$$

Eq. (36) represents the deflection function of a beam with a rigid stub of length $l - x_0$ at the right end, or analogously of a clamped–clamped beam of length x_0.

5.1.2. Slope discontinuity

The case of slope discontinuity located at x_0 is considered. The beam under study is depicted in Fig. 5 where both the flexural stiffness with a Dirac's delta and the internal hinge with rotational spring models have been reported. The two models are equivalent under the condition that the stiffness of the rotational spring is $k_\varphi = \frac{1-\gamma A}{\gamma}E_0I_0$, as discussed in Section 4, where the constant A takes the value 2.013 as indicates in Appendix A, and $\gamma = 0.49$ fulfilling inequality $0 \leqslant \gamma \leqslant 1/A$ is assumed for the numerical application.

For the beam under study the integration constants c_1, c_2, c_3, c_4, evaluated by imposing the boundary conditions, are

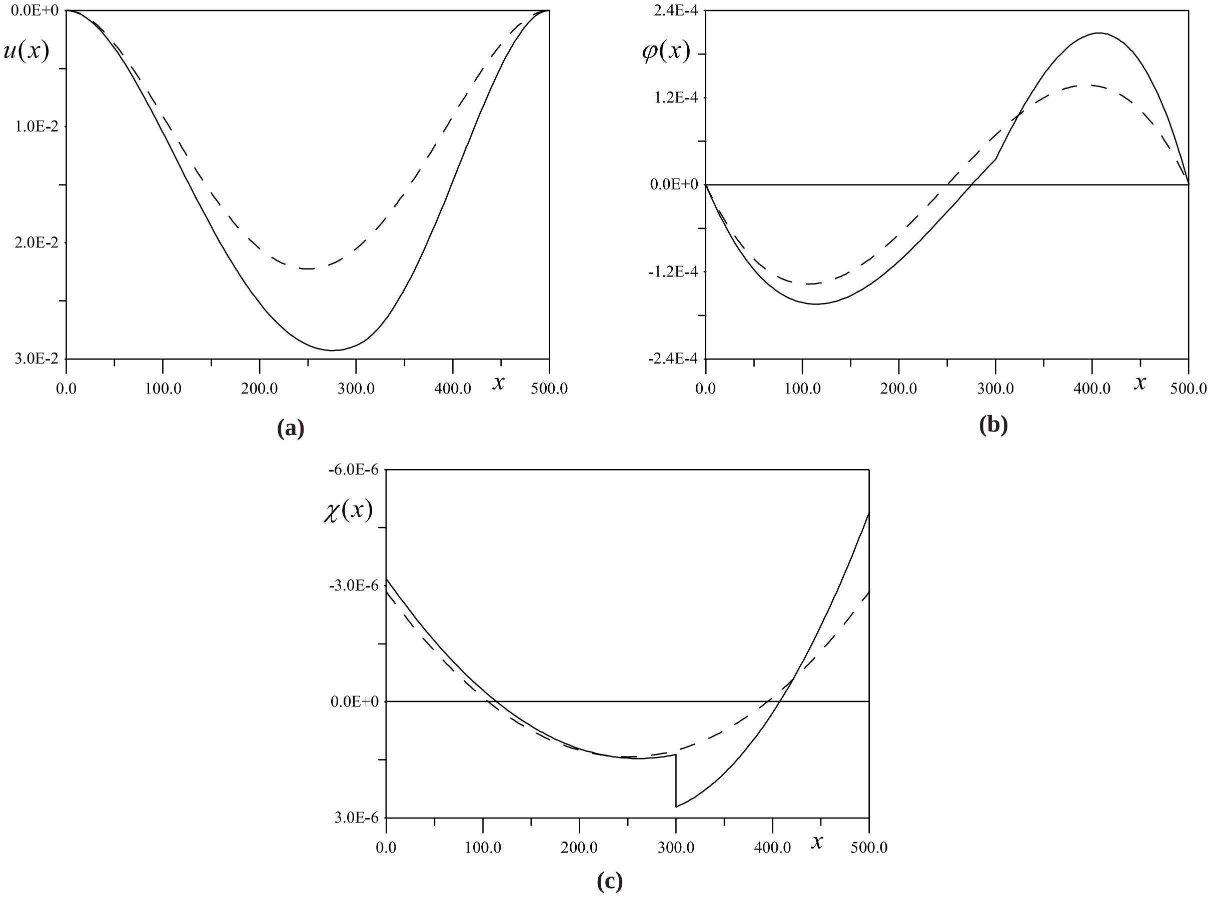

Fig. 3. Solutions of the clamped–clamped beam: (a) deflection; (b) slope; (c) curvature. (—) Beam with flexural stiffness discontinuity, (- - -) uniform beam.

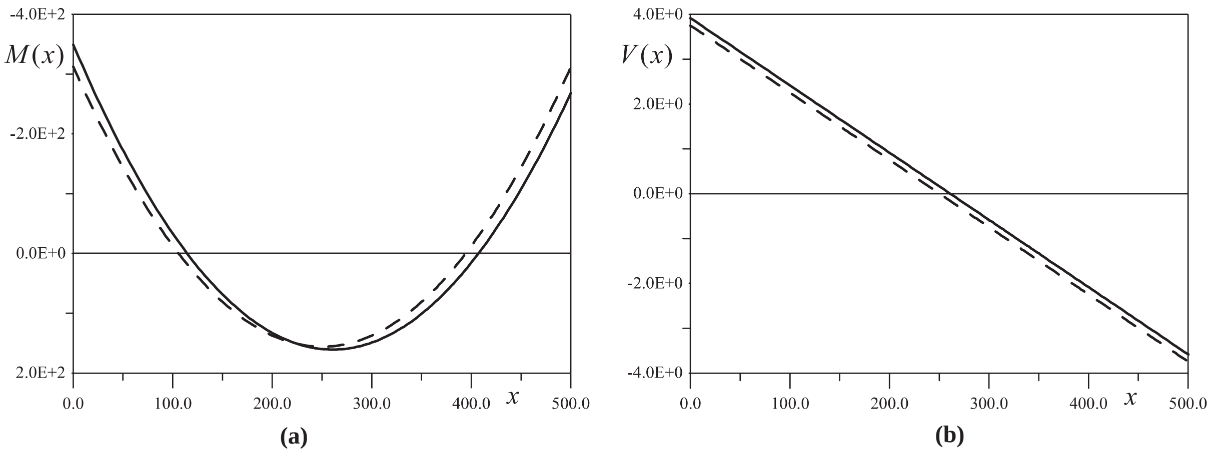

Fig. 4. Solutions of the clamped–clamped beam: (a) bending moment; (b) shear force. (—) Beam with flexural stiffness discontinuity, (- - -) uniform beam.

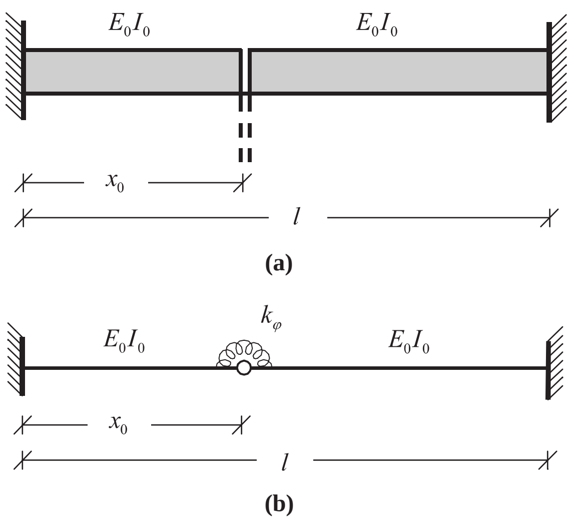

Fig. 5. (a) Clamped–clamped beam with slope discontinuity; (b) clamped–clamped beam with internal hinge and rotational spring.

$$c_1 = 0; \quad c_2 = 0; \quad c_3 = \frac{q}{24E_0I_0} \frac{(1-\gamma A)l^5 + 6\gamma x_0 l\left(3l^2 - 8x_0 l + 6x_0^2\right)}{(1-\gamma A)l^3 + 4\gamma\left(l^2 - 3x_0 l + 3x_0^2\right)} \tag{37a,b,c}$$

$$c_4 = -\frac{q}{12E_0I_0} \frac{(1-\gamma A)l^4 + \gamma\left(3l^3 - 4x_0 l^2 - 6x_0^2 l + 12x_0^3\right)}{(1-\gamma A)l^3 + 4\gamma\left(l^2 - 3x_0 l + 3x_0^2\right)} \tag{37d}$$

The closed form solutions in terms of deflection, slope and curvature functions obtained in Section 4 and given by Eqs. (27)–(29), respectively, are plotted in Fig. 6 and compared with the functions of the uniform beam. Fig. 6a shows the continuity of the deflection function with an increment with respect to the uniform beam as a result of the adopted flexural stiffness variation. Fig. 6b shows the expected discontinuity of the slope function, and Fig. 6c shows that the curvature possesses a Dirac's delta located at x_0. In Fig. 7 bending moment and shear force, given by Eqs. (30) and (31), have been plotted and are different from the uniform beam since c_3 and c_4 are influenced by the singularity at x_0, in the statically indeterminate beam, as shown by Eq. (37c,d).

5.2. Boundary discontinuity

In the presented closed form solutions the case of discontinuity located at $0 < x_0 < l$ has been considered in order to avoid any coincidence of the singularity with the boundary conditions. This circumstance might have caused the impossibility of imposing the boundary conditions to distributions centered at boundaries.

However in this section we will consider the proposed closed form solutions for x_0 moving toward the boundaries of the beam. The limit under study has to be evaluated once the boundary conditions have been imposed. It will be shown that singularities of the flexural stiffness superimposed on boundary conditions are able to modify the boundary conditions themselves.

In this section the clamped–clamped beam shown in Fig. 1 and studied in the Section 5.1 is considered.

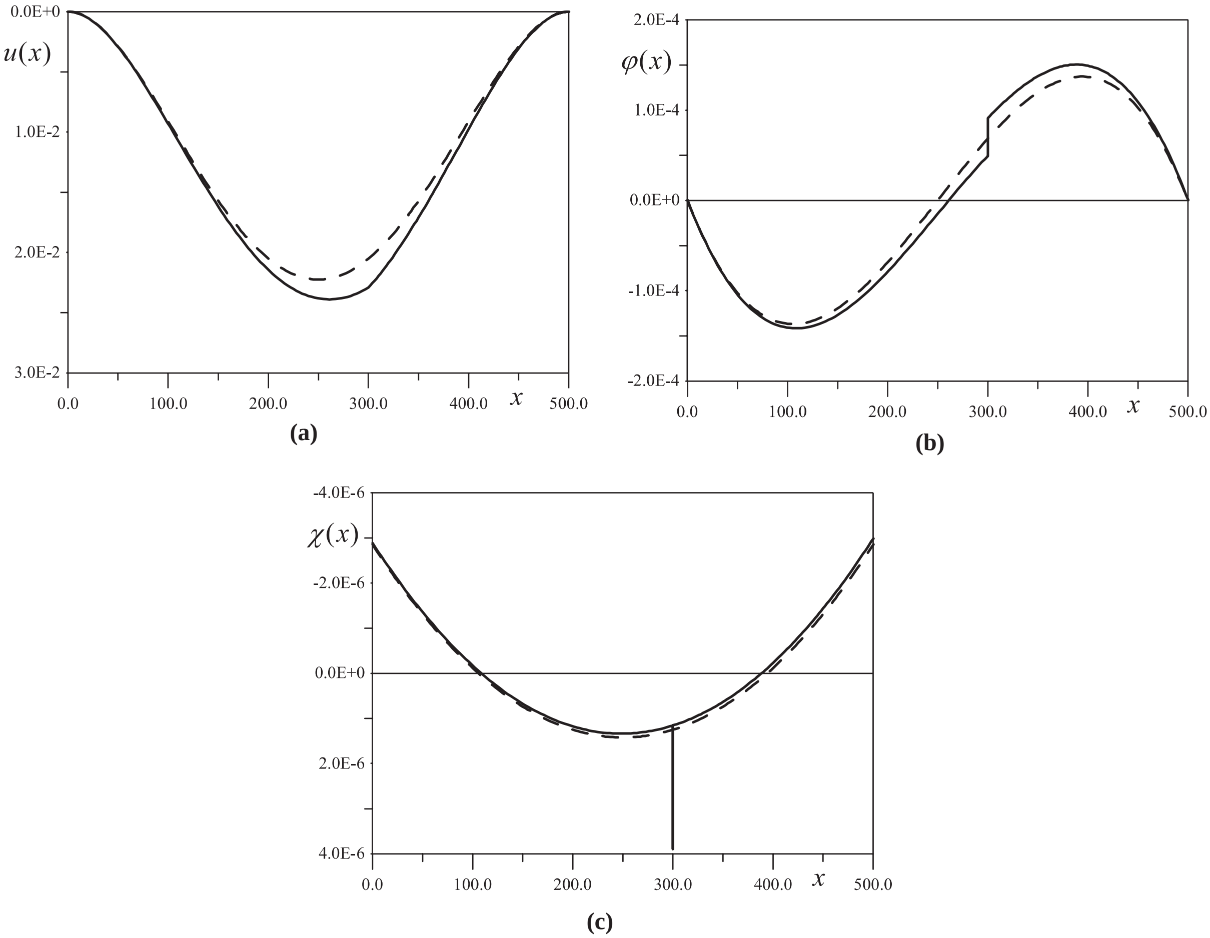

Fig. 6. Solutions of the clamped–clamped beam: (a) deflection; (b) slope; (c) curvature. (—) Beam with slope discontinuity, (- - -) uniform beam.

5.2.1. Flexural stiffness discontinuity

The case of abrupt flexural stiffness change modeled as the unit step function $\gamma U(x - x_0)$ for x_0 next to the boundary $x = l$ represents the circumstance that only a portion of the cross-section of the beam edge is clamped. Moreover the discontinuity intensity $\gamma = 1$ leads to no flexural stiffness at $x_0 = l$, which represents the case of free edge of the beam. In fact by substituting the integration constants c_1, c_2, c_3, c_4 given by Eq. (35) into the deflection function given by Eq. (20) and making the limit for $x_0 \to l$, $\forall \gamma < 1$, the following relationship is obtained:

$$u(x) = \frac{q}{24E_0I_0}(x^2 - 2lx + l^2)x^2 \tag{38}$$

which represents the deflection function of a clamped–clamped beam of length l. It follows that for $x_0 \to l$ the discontinuity intensity γ does not influence the beam, that results clamped at $x = l$. A different case is recovered for $\gamma \to 1$. In fact by substituting the integration constants given by Eq. (35) into the deflection function given by Eq. (20) and making the limits for $\gamma \to 1$ and for $x_0 \to l$ the following relationship is obtained:

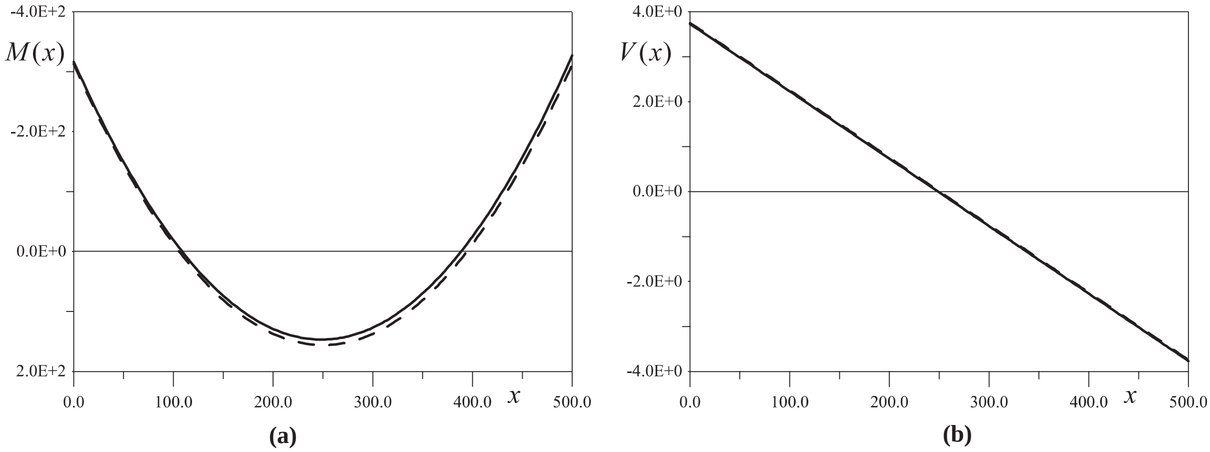

Fig. 7. Solutions of the clamped–clamped beam: (a) bending moment; (b) shear force. (—) Beam with slope discontinuity, (---) uniform beam.

$$u(x) = \frac{q}{24E_0I_0}\left[(x^2 - 4lx + 6l^2)x^2 - 3l^4U(x-l)\right] \tag{39}$$

which represents for $0 \leqslant x < l$ the deflection function of a cantilever beam of length l.

5.2.2. Slope discontinuity

The case of slope discontinuity modeled as a Dirac's delta distribution $\gamma\delta(x - x_0)$ centered at $x_0 = l$ represents the case of an external support with rotational spring with stiffness $k_\varphi = \frac{1-\gamma A}{\gamma}E_0I_0$. Moreover the value $\gamma = 1/A$ for the discontinuity intensity leads to the absence of rotational spring in the external support and the case of clamped-supported beam is recovered.

The solution of this case is obtained by substituting the integration constants c_1, c_2, c_3, c_4 given by Eq. (37) into the deflection function given by Eq. (27) and making the limits for $x_0 \to l$ and $\gamma \to 1/A$, and the following relationship is obtained:

$$u(x) = \frac{q}{48E_0I_0}\left[(2x^2 - 5lx + 3l^2)x^2 - l^3(l-x)U(x-l)\right] \tag{40}$$

For $0 \leqslant x < l$ Eq. (40) represents the deflection function of a clamped-supported beam of length l.

6. Conclusions

The problem of the integration of the fourth order static governing differential equation of the uniform Euler–Bernoulli beam in presence of singularities has been treated.

The case of discontinuous flexural stiffness has been modeled by superimposing a unit step function to a constant flexural stiffness. Superimposition of a Dirac's delta distribution to a constant flexural stiffness has been shown to lead to cases with slope discontinuities.

The governing differential equation of an Euler–Bernoulli beam showing flexural stiffness and slope discontinuities has been derived, and closed form solutions requiring only the knowledge of the four boundary conditions have been presented and discussed.

The presented integration procedure for the case of slope discontinuity requires the definition of the product of distributions, provided in the literature by means of different approaches. The choice concerning the approach adopted in the paper, leading to the solution of multiplication of two Dirac's delta distributions centered at the same point, finds its justification (proved to be desirable) since has led to the exact solution usually provided by means of approaches which do not rely on the distribution theory.

It has to be remarked that bending moment and shear force functions do not depend directly on the singularity intensity whose influence appears only through the boundary conditions. This circumstance can be explained since bending moment and shear force functions for statically determinate beams are independent of the flexural stiffness.

Furthermore, closed form expressions for bending moment and shear force functions are identical for the two types of adopted discontinuities, the differences appear for statically indeterminate beams only when boundary conditions are imposed.

Slope discontinuity modeled through the flexural stiffness by means of a Dirac's delta distribution has been shown to be correspondent to the presence of an internal hinge with rotational spring whose elastic constant, relating slope discontinuity and bending moment values at the same abscissa, has been provided explicitly as function of the flexural stiffness discontinuity intensity.

It has to be noted that the presented integration procedure can easily be applied to cases of non-uniform beams showing singularity. The governing equation can be written in terms of a single deflection function and closed form expression of the solution can be obtained for those flexural stiffness functions whose exact integration is given in the literature without singularities.

The cases of singularity centered at the boundaries has been initially neglected in order to avoid additional problems in the application of the integration rules for distributions. However it has been shown that the proposed closed form solutions are able to account for cases of singularities centered at the boundaries provided that the boundary conditions are imposed first. Hence, if boundary conditions of a clamped–clamped beam are considered first, singularities at boundaries are able to release the external constraints. The latter circumstance suggests the treatment of a clamped–clamped beam and to account for the real external constraints as singularities of the flexural stiffness centered at boundary. Such a procedure should lead to elimination of the solution dependence on the boundary conditions.

It must be pointed out that the integration procedure has been presented for cases with a single singularity in order to provide the physical interpretation of the proposed stiffness models. The computational effort required by traditional methods relying on the imposition of the continuity conditions and avoided by the proposed closed form solutions, although not remarkable for the case of a single singularity, represents a necessary basis for the development of the case of multiple singularities, possibly of different types.

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Appendix 1. Definition of the product of two Dirac's deltas

The adoption of singularities in the flexural stiffness of an Euler–Bernoulli beam modeled as a Dirac's delta requires in this paper ad hoc considerations in order to provide physical evidence. In particular it has been shown in Section 2 haw such a case requires the introduction of the product of two Dirac's deltas (Eq. (10)).

In the classical theory of distributions although the product of distributions is well defined the proposed definitions cannot be extended to he product of two Dirac's deltas centered at the same point.

In the literature some theories addressed the problem of definition of new classes of multiplication of distributions to be applied to two or more Dirac's deltas centered at the same point.

Usually, theories attempting a definition of the product of distributions rely on: (i) regularization of the distributions in order to obtain continuous functions able to return to the original distributions by means of a limiting procedure; (ii) multiplication, in the sense of distributions, of the regularized distributions; (iii) definition of the product of two or more distributions by means of a limiting procedure applied to the multiplication of the regularized distributions as defined in step (ii). The theory proposed by Bremermann and Durand (1961) is based on a regularization of distributions by means of the so called analytic continuation of a distribution. The Colombeau's theory (Colombeau, 1984) follows a different approach to define a regularized version of a distribution, called *of the sequential completion*. The latter makes use of the so called δ-sequences, and the regularized distribution is defied as the convolution of the original distribution with the δ-sequences. It has to be remarked that the above mentioned theories for regularized distributions do not allow the definition of the product of Dirac's deltas.

In this appendix a different approach, proposed by Bagarello (1995, 2002), which makes use of both the above mentioned definitions of regularized distributions in order to introduce a new multiplication for distributions is reported. The multiplication introduced by Bagarello applies only to distributions for which both analytic continuation, dependent on a α parameter, and convolution with δ-sequences, dependent on a β parameter, exist, and it has been proved to apply to Dirac's delta and its derivatives. In particular, according to Bagarello (1995), a regularized distribution δ_{red} of a Dirac's delta is considered first by means of an analytic continuation as follows:

$$\delta_{\text{red}}\left(x,\frac{1}{n^{\alpha}}\right)=\frac{1}{\pi n^{\alpha}}\frac{1}{x^{2}+\frac{1}{n^{2\alpha}}} \tag{A1.1}$$

and then another regularized distribution $\delta_n^{(\beta)}$ is considered by means of the following δ-sequence:

$$\delta_n^{(\beta)}(x)=n^{\beta}\Phi(n^{\beta}x) \tag{A1.2}$$

for any fixed n and where $\Phi(x)$ is a suitable chosen function with support $[-1,1]$ and such that $\int_{-1}^{1}\Phi(x)\,\mathrm{d}x=1$.

According to the multiplication for distributions proposed by Bagarello (1995) the product of two Dirac's deltas, making use of the regularized distributions reported in Eqs. (A1.1) and (A1.2) depending on the choice of the parameters α and β, is defined as follows:

$$(\delta(x)\delta(x))_{\alpha,\beta}(\Psi(x))=\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n^{(\beta)}(x)\delta_{\text{red}}\left(x,\frac{1}{n^{\alpha}}\right)\Psi(x)\,\mathrm{d}x \tag{A1.3}$$

for any test function $\Psi(x)$.

The limit of the sequence defined in Eq. (A1.3) exists if we require the function $\Phi(x)$ appearing in Eq. (A1.2) to be of the form:

$$\Phi(x)=\begin{cases}\frac{x^{m}}{F}\exp\left\{\frac{1}{x^{2}-1}\right\} & |\,x\,|<1\\ 0 & |\,x\,|\geqslant 1\end{cases} \tag{A1.4}$$

with m a natural number and F a normalization constant and the fulfillment of the inequality:

$$\alpha-2\beta\geqslant 0 \tag{A1.5}$$

The limit of the sequence in Eq. (A1.3) under the conditions provided by Eqs. (A1.4) and (A1.5) defines the product of two Dirac's deltas as follows (Bagarello, 1995, 2002):

$$(\delta(x)\delta(x))_{\alpha,\beta}(\Psi(x)) = \begin{cases} A_j\delta(x)(\Psi(x)) & \alpha = 2\beta \\ 0 & \alpha > 2\beta \end{cases} \tag{A1.6}$$

where

$$A_j = \frac{1}{\pi}\int_{-1}^{1} \frac{\Phi(x)}{x^j}\,\mathrm{d}x \tag{A1.7}$$

In this paper we adopt the first option provided by Eq. (A1.6) as the product of two Dirac's deltas which returns the properties of a single Dirac's delta if $\alpha = 2\beta$ is assumed. Furthermore, in order to guarantee the existence of the integral in Eq. (A1.7) it is assumed $j = 2$ and $m = 2$ appearing in Eq. (A1.4) such that:

$$A_2 = \frac{1}{\pi}\int_{-1}^{1} \frac{\Phi(x)}{x^2}\,\mathrm{d}x = 2.013 \tag{A1.8}$$

According to Eq. (A1.6) for $\alpha = 2\beta$ the product of two Dirac's deltas both centered at x_0 is a single Dirac's delta and will be adopted throughout the paper by means of the following formal expression:

$$\delta(x - x_0)\delta(x - x_0) = A\delta(x - x_0) \tag{A1.9}$$

where the application of the Dirac's delta to any test function is implicitly assumed and where the constant $A = A_2 = 2.013$ defined by Eq. (A1.8) is adopted.

Appendix 2. Integration procedure of the fourth order governing differential equation

In this section an integration procedure of the fourth order governing differential Eq. (23) for the singular flexural stiffness expressed by Eq. (8) is presented.

The differential Eq. (23) can be rewritten as follows:

$$E_0I_0\left[(1 - \gamma\delta(x - x_0))u^{\mathrm{IV}}(x) - 2\gamma\delta^{\mathrm{I}}(x - x_0)u^{\mathrm{III}}(x) - \gamma\delta^{\mathrm{II}}(x - x_0)u^{\mathrm{II}}(x)\right] = q(x) \tag{A2.1}$$

Integration of Eq. (A2.1) will be performed by considering all terms containing the Dirac's delta distribution $\delta(x - x_0)$ and their derivatives as additional loading terms acting on a uniform beam as follows:

$$E_0I_0u^{\mathrm{IV}}(x) = \bar{q}(x) \tag{A2.2}$$

where the augmented loading function $\bar{q}(x)$ is defined as

$$\bar{q}(x) = q(x) + \gamma E_0I_0\left[\delta(x - x_0)u^{\mathrm{IV}}(x) + 2\delta^{\mathrm{I}}(x - x_0)u^{\mathrm{III}}(x) + \delta^{\mathrm{II}}(x - x_0)u^{\mathrm{II}}(x)\right] \tag{A2.3}$$

Solution of the non-homogeneous differential Eq. (A2.2) is sought under the form:

$$u(x) = u_h(x) + u_p(x) \tag{A2.4}$$

where $u_h(x)$ is the solution of the homogeneous equation associated to Eq. (A2.2) hence given as

$$u_h(x) = c_1 + c_2x + c_3x^2 + c_4x^3 \tag{A2.5}$$

by means of the introduction of the four integration constants c_1, c_2, c_3, c_4 to be determined by imposing the kinematic and mechanic boundary conditions in the solution given by Eq. (A2.4), and where $u_p(x)$ is a particular integral of Eq. (A2.2) sought under the form:

$$u_p(x) = d_1(x) + d_2(x)x + d_3(x)x^2 + d_4(x)x^3 \tag{A2.6}$$

where functions $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$ are to be found by replacing Eq. (A2.6) and its derivatives into Eq. (A2.2). Obviously, evaluation of $u_p^{\mathrm{I}}(x)$, $u_p^{\mathrm{II}}(x)$, $u_p^{\mathrm{III}}(x)$, $u_p^{\mathrm{IV}}(x)$ and substitution in the non-homogeneous Eq.

(A2.2) provides only one condition, hence evaluation of functions $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$ requires three additional conditions as indicated in what follows.

The first derivative of Eq. (A2.6) gives:

$$u_p^{\mathrm{I}}(x) = d_1^{\mathrm{I}}(x) + d_2^{\mathrm{I}}(x)x + d_3^{\mathrm{I}}(x)x^2 + d_4^{\mathrm{I}}(x)x^3 + d_2(x) + 2d_3(x)x + 3d_4(x)x^2 \tag{A2.7}$$

The choice of the first additional condition to be imposed involves the first derivatives of functions $d_1(x)$, $d_2(x)$, $d_3(x)$, d_4 (x) appearing in Eq. (A2.7) as follows:

$$d_1^{\mathrm{I}}(x) + d_2^{\mathrm{I}}(x)x + d_3^{\mathrm{I}}(x)x^2 + d_4^{\mathrm{I}}(x)x^3 = 0 \tag{A2.8}$$

Accounting for Eq. (A2.8) leads to the following constrained form for Eq. (A2.7):

$$u_p^{\mathrm{I}}(x) = d_2(x) + 2d_3(x)x + 3d_4(x)x^2; \quad \text{s.t.} \quad d_1^{\mathrm{I}}(x) + d_2^{\mathrm{I}}(x)x + d_3^{\mathrm{I}}(x)x^2 + d_4^{\mathrm{I}}(x)x^3 = 0 \tag{A2.9}$$

which does not involve derivatives of $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$. In view of Eq. (A2.9) and introducing further conditions involving the first derivatives of $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$, further derivatives $u_p^{\mathrm{II}}(x)$, $u_p^{\mathrm{III}}(x)$ can be obtained in constrained form as follows:

$$u_p^{\mathrm{II}}(x) = 2d_3(x) + 6d_4(x)x; \quad \text{s.t.} \quad d_2^{\mathrm{I}}(x) + 2d_3^{\mathrm{I}}(x)x + 3d_4^{\mathrm{I}}(x)x^2 = 0 \tag{A2.10}$$

$$u_p^{\mathrm{III}}(x) = 6d_4(x); \quad \text{s.t.} \quad 2d_3^{\mathrm{I}}(x) + 6d_4^{\mathrm{I}}(x)x = 0 \tag{A2.11}$$

and finally $u_p^{\mathrm{IV}}(x)$ is given as

$$u_p^{\mathrm{IV}}(x) = 6d_4^{\mathrm{I}}(x) \tag{A2.12}$$

In order to find the unknown functions $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$ appearing in the particular integral $u_p(x)$, the conditions constraining Eqs. (A2.9)–(A2.11) together with substitution of Eq. (A2.12) into Eq. (A2.2) have to be treated as a first order differential system given as follows:

$$\begin{cases} d_1^{\mathrm{I}}(x) + d_2^{\mathrm{I}}(x)x + d_3^{\mathrm{I}}(x)x^2 + d_4^{\mathrm{I}}(x)x^3 = 0 \\ d_2^{\mathrm{I}}(x) + 2d_3^{\mathrm{I}}(x)x + 3d_4^{\mathrm{I}}(x)x^2 = 0 \\ 2d_3^{\mathrm{I}}(x) + 6d_4^{\mathrm{I}}(x)x = 0 \\ 6EI_0 d_4^{\mathrm{I}}(x) = \bar{q}(x) \end{cases} \tag{A2.13}$$

Solution of the system given by Eq. (A2.13) in terms of $d_1^{\mathrm{I}}(x)$, $d_2^{\mathrm{I}}(x)$, $d_3^{\mathrm{I}}(x)$, $d_4^{\mathrm{I}}(x)$, in view of the definition of the augmented loading function $\bar{q}(x)$ provided by Eq. (A2.3), leads to

$$d_1^{\mathrm{I}}(x) = -\frac{q(x)x^3}{6E_0I_0} - \frac{\gamma}{6}x^3\left[u^{\mathrm{IV}}(x)\delta(x-x_0) + 2u^{\mathrm{III}}(x)\delta^{\mathrm{I}}(x-x_0) + u^{\mathrm{II}}(x)\delta^{\mathrm{II}}(x-x_0)\right] \tag{A2.14a}$$

$$d_2^{\mathrm{I}}(x) = \frac{q(x)x^2}{2E_0I_0} + \frac{\gamma}{2}x^2\left[u^{\mathrm{IV}}(x)\delta(x-x_0) + 2u^{\mathrm{III}}(x)\delta^{\mathrm{I}}(x-x_0) + u^{\mathrm{II}}(x)\delta^{\mathrm{II}}(x-x_0)\right] \tag{A2.14b}$$

$$d_3^{\mathrm{I}}(x) = -\frac{q(x)x}{2E_0I_0} - \frac{\gamma}{2}x\left[u^{\mathrm{IV}}(x)\delta(x-x_0) + 2u^{\mathrm{III}}(x)\delta^{\mathrm{I}}(x-x_0) + u^{\mathrm{II}}(x)\delta^{\mathrm{II}}(x-x_0)\right] \tag{A2.14c}$$

$$d_4^{\mathrm{I}}(x) = \frac{q(x)}{6E_0I_0} + \frac{\gamma}{6}\left[u^{\mathrm{IV}}(x)\delta(x-x_0) + 2u^{\mathrm{III}}(x)\delta^{\mathrm{I}}(x-x_0) + u^{\mathrm{II}}(x)\delta^{\mathrm{II}}(x-x_0)\right] \tag{A2.14d}$$

providing uncoupled equations for the derivatives of the unknown functions $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$.

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